# Heat kernel estimates for symmetric pure jump Dirichlet forms

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This is based on a joint work with Zhen-Qing Chen and Takashi Kumagai

The 15th Workshop on Markov Processes and Related Topics

Jilin University, July 11, 2019

# Outline

1. **Introduction**
   - Setting and history
   - Motivation

2. **Main results**
   - Heat kernel estimates
   - Remarks

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# Symmetric pure jump Dirichlet form and heat kernel

- Let $(M, d, \mu)$ be a *metric measure space*.
- Consider a regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M; \mu)$ as follows:

$$D(f, g) = \iint_{M \times M} (f(x) - f(y))(g(x) - g(y))\, J(dx, dy),$$

where $J(\cdot, \cdot)$ is a symmetric Radon measure $M \times M$.

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$$\mathbb{E}^x f(X_t) = P_t f(x) = \int p(t, x, y) f(y)\, \mu(dy), \quad x \in M_0, f \in L^\infty(M; \mu).$$

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# Symmetric $\alpha$-stable-like process: $0 < \alpha < 2$

- Symmetric $\alpha$-stable-like process (Chen-Kumagai ('03)) – This corresponds to div. form.

$$D(f,f) = \iint_{M\times M} (u(x) - u(y))^2 J(x,y)\, \mu(dx)\, \mu(dy)$$

on a $d$-set $M \subset \mathbb{R}^n$, where J($\alpha$) holds, i.e.,

$$J(x,y) \asymp \frac{1}{d(x,y)^{d+\alpha}}$$

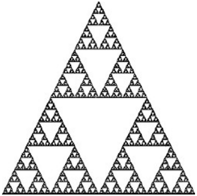
with $0 < \alpha < 2$.

- HK($\alpha$):

$$c_1 \left( t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}} \right) \leqslant p(t,x,y) \leqslant c_2 \left( t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}} \right)$$

for all $t > 0$ and $x, y \in M$.

# Symmetric $\alpha$-stable-like process: $\alpha > 2$



## Question

*Let $M$ be a Sierpinski gasket on $\mathbb{R}^2$, and $\mu(B(x,r)) \asymp r^d$ with $d = \frac{\log 3}{\log 2}$. Let*

$$D(f,f) \asymp \iint_{M\times M} \frac{(f(x)-f(y))^2}{d(x,y)^{d+\alpha}}\,\mu(dx)\,\mu(dy),$$

*where $\alpha \in (0, \frac{\log 5}{\log 2})$ (possibly $\alpha \geqslant 2$). What is the expression of HK?*

$$p(t,x,y) \asymp \frac{1}{t^{d/\alpha}} \wedge \frac{t}{d(x,y)^{d+\alpha}} ???$$

*Yes! see Chen-Kumagai-W., ('18).*

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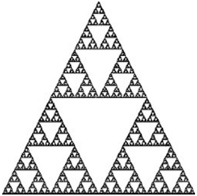

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# Jumping kernel

- $$\mathcal{E}(f,g) = \iint_{M\times M} (f(x)-f(y))(g(x)-g(y))\, J(dx,dy).$$

- $J_\phi$:

$$J(x,y) \asymp \frac{1}{V(x,d(x,y))\phi(d(x,y))}.$$

- Example: symmetric $\alpha$-stable processes on $\mathbb{R}^d$

$$J(x,y) \asymp \frac{1}{|x-y|^{d+\alpha}}.$$

- MMS $(M,d,\mu)$. Let $V(x,r)=\mu(B(x,r))$ for all $x\in M$ and $r>0$. VD and RVD hold:

$$c_1\left(\frac{R}{r}\right)^{d_1} \leqslant \frac{V(x,R)}{V(x,r)} \leqslant c_2\left(\frac{R}{r}\right)^{d_2}, \quad x\in M, 0<r<R.$$

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- Example: $[\alpha_1,\alpha_2] \subset (0,\infty)$, $\nu$ is a finite measure on $[\alpha_1,\alpha_2]$ such that

$$\phi(r) := \left(\int_{\alpha_1}^{\alpha_2} r^{-\alpha}\,\nu(d\alpha)\right)^{-1}.$$

Especially, $\mu(B(x,r)) \asymp r^d$, $0<\alpha_1<\cdots<\alpha_n<\infty$,

$$J(x,y) = \sum_{k=1}^{n} \frac{c_i(x,y)}{d(x,y)^{d+\alpha_i}},$$

where $c^{-1} \leqslant c_i(x,y) = c_i(y,x) \leqslant c$ (mixed-type process).

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$$c_3\left(\frac{R}{r}\right)^{\beta_1} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant c_4\left(\frac{R}{r}\right)^{\beta_2}, \quad 0 < r < R.$$

# Heat kernel estimates

- $HK(\phi)$:

$$p(t,x,y) \asymp \frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}.$$

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# Symmetric $\alpha$-stable-like process: $0 < \alpha < \infty$

**Theorem (Chen-Kumagai-W., '18)**

*The following are equivalent:*

(i) $HK(\phi)$.

(ii) $J_\phi$ *and* $CS(\phi)$.

- $CS(\phi)$: For $0 < r \leqslant R, f \in \mathcal{F}$ and almost all $x \in M$, there exists a cutoff function $\varphi$ for $B(x,R) \subset B(x,R+r)$ so that the following holds:

$$\int_{U^*} f^2 \, d\Gamma(\varphi,\varphi) \leqslant C_1 \int_{U \times U^*} (f(x)-f(y))^2 J(dx,dy) + \frac{C_2}{\phi(r)} \int_{U^*} f^2 \, d\mu,$$

  where $U^* = B(x, R + (1+C_0)r) \setminus B(x, R - C_0 r)$, $C_0 \in (0,1]$ and $U = B(x,R+r) \setminus B(x,R)$.

- $CS(\phi)$ holds trivially for the case that $\phi(r) = r^\alpha$ with $\alpha \in (0,2)$.

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# Counterexample: $HK(\phi)$

**Example ($J_\phi$ alone does not imply $HK(\phi)$.)**

Let $M = \mathbb{R}^d$, $\phi(r) = r^\alpha \vee r^\beta$ with $0 < \alpha < 2 < \beta$, and

$$J(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|)}, \quad x, y \in \mathbb{R}^d.$$

Then, $J_\phi$ holds, but $HK(\phi)$ does not hold.

- $CS(\phi_*)$ holds with $\phi_*(r) \asymp r^\alpha \vee r^2$.
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$$HK(\phi) \Longleftrightarrow J_\phi + CS(\phi).$$
- Question: Can we give a similar example in the setting MMS? (asked by Professor Grigor'yan.) Yes!

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# New counterexample: $HK(\phi)$

**Example**

Suppose $X := \{X_t, t \geqslant 0; \mathbb{P}^x, x \in M\}$ is a conservative symmetric diffusion process on $M$ such that

$$q(t,x,y) \asymp \frac{1}{V(x,t^{1/\beta})} \exp\left(-c\left(\frac{d(x,y)^\beta}{t}\right)^{1/(1-\beta)}\right), \quad t > 0, x, y \in M$$

for some $\beta \geqslant 2$. (Sub-Gaussian estimates, see Barlow-Perkins, $\cdots$)

Let $S := (S_t)_{t\geqslant 0}$ be a subordinator with $S_0 = 0$ that is independent of $X$ and has the Laplace exponent

$$f(r) = \int_0^\infty (1 - e^{-rs})\nu(s)\,ds, \quad r > 0,$$

where

$$\nu(s) = \frac{1}{s^{1+\gamma_1}} 1_{\{0<s\leqslant 1\}} + \frac{1}{s^{1+\gamma_2}} 1_{\{s>1\}}$$

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Let $Y := (Y_t)_{t\geqslant 0}$ be the subordinate process defined by $Y_t := X_{S_t}$ for all $t > 0$.

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Then, $Y$ is a symmetric jump process such that

(i) its jumping kernel $J(dx, dy)$ has a density with respect to the product measure $\mu \times \mu$ given by

$$J(x,y) \simeq \begin{cases} \frac{1}{V(x,d(x,y))d(x,y)^{\alpha}} & d(x,y) \leqslant 1, \\ \frac{1}{V(x,d(x,y))d(x,y)^{\alpha'}} & d(x,y) \geqslant 1, \end{cases}$$

where $\alpha = \gamma_1 \beta$ and $\alpha' = \gamma_2 \beta$.

(ii) for any $x \in M$ and $r > 0$,

$$\mathbb{E}^x \left[ \tau^Y_{B(x,r)} \right] \simeq r^{\alpha} \vee r^{\beta}.$$

An interesting point: $\alpha < \beta < \alpha'$, and the scaling of the jumping kernel $\phi(r) := r^{\alpha} \vee r^{\alpha'}$ is different from the scaling of the process $\phi_*(r) = r^{\alpha} \vee r^{\beta}$!

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# Setting

- $$\mathcal{E}(f,g) = \iint_{M\times M} (f(x)-f(y))(g(x)-g(y))\, J(dx,dy).$$

- $J_{\phi_j}$:
$$J(x,y) \asymp \frac{1}{V(x,d(x,y))\phi_j(d(x,y))}.$$

- $$c_1\left(\frac{R}{r}\right)^{\beta_{1,\phi_j}} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant c_2\left(\frac{R}{r}\right)^{\beta_{1,\phi_j}}, \quad 0<r<R.$$

- $$p^{(j)}(t,x,y) = \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi_j(d(x,y))}.$$

- Note: $p^{(j)}(t,x,y)$ should not be the expression of the HK!

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- Diffusive-like scaling function $\phi_c(r)$:

$$c_3\left(\frac{R}{r}\right)^{\beta_{1,\phi_c}} \leqslant \frac{\phi_c(R)}{\phi_c(r)} \leqslant c_4\left(\frac{R}{r}\right)^{\beta_{2,\phi_c}} \quad \text{for all } 0<r\leqslant R.$$

- Suppose that there is a constant $c_0 \geqslant 1$ such that

$$\phi_c(r) \leqslant c_0\phi_j(r) \quad \text{for all } r \geqslant 0.$$

For instance, for the process $Y$ in the example above, $\phi_j(r) = r^\alpha \vee r^{\alpha'}$ and $\phi_c(r) = r^\beta$ (sub-Gaussian estimates) with $\alpha < \beta < \alpha'$.

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$$c_3\left(\frac{R}{r}\right)^{\beta_{1,\phi_c}} \leqslant \frac{\phi_c(R)}{\phi_c(r)} \leqslant c_4\left(\frac{R}{r}\right)^{\beta_{2,\phi_c}} \quad \text{for all } 0 < r \leqslant R.$$

- Suppose that there is a constant $c_0 \geqslant 1$ such that
$$\phi_c(r) \leqslant c_0\phi_j(r) \quad \text{for all } r \geqslant 0.$$

For instance, for the process $Y$ in the example above, $\phi_j(r) = r^\alpha \vee r^{\alpha'}$ and $\phi_c(r) = r^\beta$ (sub-Gaussian estimates) with $\alpha < \beta < \alpha'$.

# Setting

- $$\mathcal{E}(f,g)=\iint_{M\times M}(f(x)-f(y))(g(x)-g(y))\,J(dx,dy).$$

- $J_{\phi_j}$:
$$J(x,y)\asymp\frac{1}{V(x,d(x,y))\phi_j(d(x,y))}.$$

- Diffusive-like scaling function $\phi_c(r)$:
$$c_3\left(\frac{R}{r}\right)^{\beta_{1,\phi_c}}\leqslant\frac{\phi_c(R)}{\phi_c(r)}\leqslant c_4\left(\frac{R}{r}\right)^{\beta_{2,\phi_c}}\quad\text{for all } 0<r\leqslant R.$$

- Suppose that there is a constant $c_0\geqslant 1$ such that
$$\phi_c(r)\leqslant c_0\phi_j(r)\quad\text{for all } r\geqslant 0.$$

For instance, for the process $Y$ in the example above, $\phi_j(r)=r^\alpha\vee r^{\alpha'}$ and $\phi_c(r)=r^\beta$ (sub-Gaussian estimates) with $\alpha<\beta<\alpha'$.

# Main result: heat kernel estimates

**Theorem (Chen-Kumagai-W., '19+)**

*Assume that the metric measure space $(M, d, \mu)$ satisfies VD and RVD, and that $\phi_c, \phi_j$ satisfy the weak scaling property. Then the following are equivalent:*

(1) $HK(\phi_j, \phi_c)$.

(2) $PI(\phi)$, $J_{\phi_j}$ *and* $CS(\phi)$,

*where*

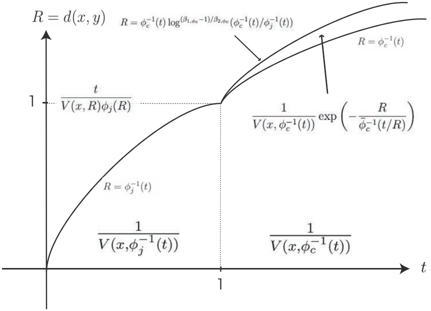
$$\phi(r) := \begin{cases} \phi_j(r), & r \in (0, 1], \\ \phi_c(r), & r \in [1, \infty). \end{cases}$$

- For instance, for the process $Y$ in the example above, $\phi_j(r) = r^\alpha \vee r^{\alpha'}$ (the scaling function of jumping kernel) and $\phi(r) = r^\alpha \vee r^\beta$ (the scaling function of the process) with $\alpha < \beta < \alpha'$.
- $PI(\phi)$:

$$\int_{B_r} (f - \bar{f}_{B_r})^2 \, d\mu \leqslant C\phi(r) \int_{B_r \times B_r} (f(x) - f(y))^2 \, J(dx, dy).$$

# Heat kernel estimates

- $HK(\phi_j, \phi_c)$ :




- $$p^{(j)}(t,x,y) = \frac{1}{V(x, \phi_j^{-1}(t))} \wedge \frac{t}{V(x, d(x,y))\phi_j(d(x,y))}.$$

- For example, in $\mathbb{R}^d$, if $V(x,r) = r^d$, $\phi_c(r) = r^2$ and $d(x,y) = |x-y|$, then the first two terms for $t \geqslant 1$ are the standard Gaussian estimate.

# Heat kernel estimates

- $HK(\phi_j, \phi_c)$:

$$p(t,x,y) \asymp \begin{cases} p^{(j)}(t,x,y), & 0 < t \leqslant 1, \\ \frac{1}{V(x,\phi_c^{-1}(t))} \wedge \left(p^{(c)}(t,x,y) + p^{(j)}(t,x,y)\right), & t \geqslant 1, \end{cases}$$

where

$$p^{(j)}(t,x,y) := \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi_j(d(x,y))}$$

and

$$p^{(c)}(t,x,y) := \frac{1}{V(x,\phi_c^{-1}(t))} \exp\left(-\frac{d(x,y)}{\bar{\phi}_c^{-1}(t/d(x,y))}\right)$$

with

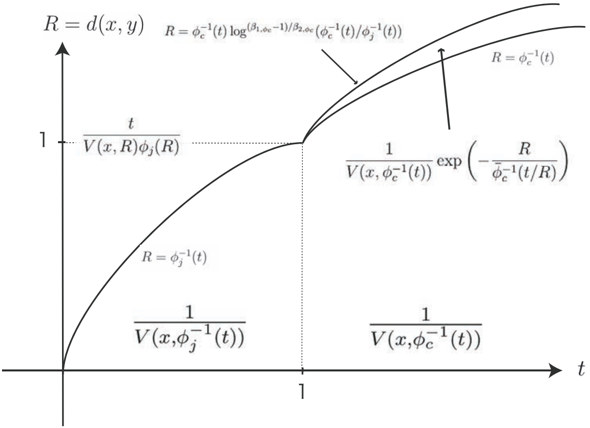
$$\bar{\phi}_c(r) \asymp \phi_c(r)/r \quad \text{for all } r > 0.$$

- For example, in $\mathbb{R}^d$, if $V(x,r) = r^d$, $\phi_c(r) = r^2$ and $d(x,y) = |x-y|$, then $p^{(c)}(t,x,y)$ is the standard Gaussian estimate.

# Outline

1. Introduction
   - Setting and history
   - Motivation
2. Main results
   - Heat kernel estimates
   - Remarks

- $HK(\phi_j, \phi_c)$ :


$R = d(x,y)$
$R = \phi_c^{-1}(t) \log^{(\beta_{1,\phi_c}-1)/\beta_{2,\phi_c}}(\phi_c^{-1}(t)/\phi_j^{-1}(t))$
$R = \phi_c^{-1}(t)$
1
$\frac{t}{V(x,R)\phi_j(R)}$
$\frac{1}{V(x,\phi_c^{-1}(t))} \exp\left(-\frac{R}{\bar{\phi}_c^{-1}(t/R)}\right)$
$R = \phi_j^{-1}(t)$
$\frac{1}{V(x,\phi_j^{-1}(t))}$
$\frac{1}{V(x,\phi_c^{-1}(t))}$
1
$t$

# Heat kernel estimates for diffusions with jumps

- Consider a regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M;\mu)$ as follows:

$$\begin{aligned} D(f,g) =& \mathcal{E}^{(c)}(f,g) + \iint_{M\times M} (f(x)-f(y))(g(x)-g(y))\, J(dx,dy) \\ =:& \mathcal{E}^{(c)}(f,g) + \mathcal{E}^{(j)}(f,g), \end{aligned}$$

  where $(\mathcal{E}^{(c)}, \mathcal{F})$ is the strongly local part of $(\mathcal{E}, \mathcal{F})$, and $J(\cdot,\cdot)$ is a symmetric Radon measure $M \times M$.

- Song-Vondracek ('07): the mixture of Brownian motions and symmetric $\alpha$-stable processes on $\mathbb{R}^d$:

$$\Delta + \Delta^{\alpha/2}.$$

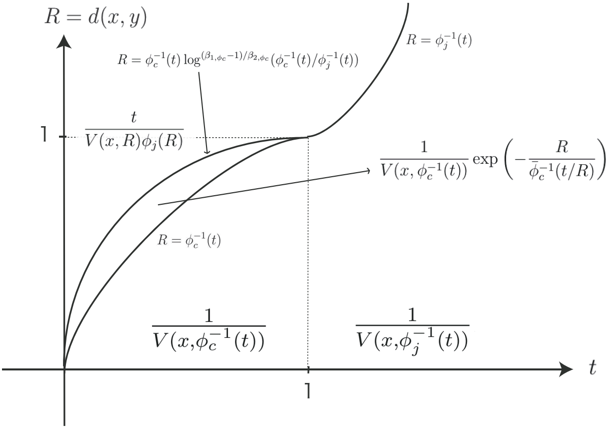
- Chen-Kugamai ('10): General diffusions with jumps:

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f(x)\cdot A(x)\nabla f(x)\,dx + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x)-f(y)^2)}{|x-y|^{d+\alpha}} c(x,y)\,dx\,dy.$$

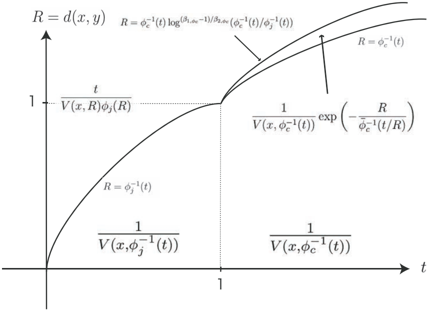
# Heat kernel estimates for diffusions with jumps

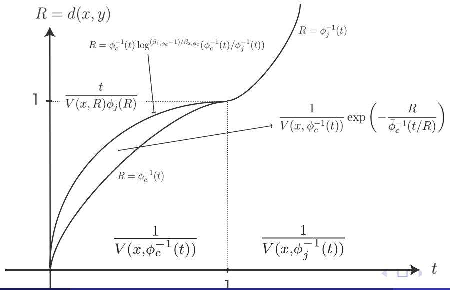
- Chen-Kugamai-W. ('19):

$$p(t,x,y) \asymp \frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \left(p^{(c)}(t,x,y) + p^{(j)}(t,x,y)\right).$$



# HK($\phi_j, \phi_c$) and heat kernel estimates for diffusions with jumps



# Remark 2: Diffusive-like scaling function $\phi_c(r)$

- In the setting of $\mathbb{R}^d$, we usually take $\phi_c(r) = r^2$ for all $r > 0$; but this is not always true.
- For example, when $\phi_j(r) = r^\alpha \vee r^2$ for all $r > 0$ with $\alpha \in (0, 2)$, then

$$\phi_c(r) \asymp r^\alpha 1_{\{0<r\leqslant 1\}} + \frac{r^2}{\log(1+r)} 1_{\{r>1\}}$$

for all $r > 0$. Then, for any $t \geqslant 1$ and $x, y \in \mathbb{R}^d$,

$$p^{(c)}(t,x,y) \asymp \frac{1}{V(x,(t\log(1+t))^{1/2})} \exp\left(-\frac{d(x,y)^2}{t\log(1+t/d(x,y))}\right).$$

This does not belong to the so-called (sub)-Gaussian estimates. Indeed long time behavior of the process is super-diffusive, and the estimates are "super-Gaussian".

- Panki's talk: Heat kernel estimates for symmetric jump processes with general mixed polynomial growths, July 14, 17:00–17:30.

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*Thank you for your attention!*